Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On Sylvester's Proof of the Reality of the Roots of Lagrange's Determinantal Equation.

BY THOMAS MUIR, LL. D.

(1). Of the various proofs that the roots of the equation

$$\begin{vmatrix} a_1-x & a_2 & a_3 & \dots \\ b_1 & b_2-x & b_3 & \dots \\ c_1 & c_2 & c_3-x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

are all real when the determinant is axisymmetric none is more elegant than Sylvester's, which appeared originally in the Philosophical Magazine, IV, pp. 138–142 (1852). In this account it is desirable to examine whether it be not readily applicable to prove the extension of the theorem which has been recently formulated,* viz. that

The n^{thic} *equation*

$$\begin{vmatrix} 11-x & 12 & 13 & \dots \\ 21 & 22-x & 23 & \dots \\ 31 & 32 & 33-x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

will have all its roots real if in the case of every pair μ, ν *of the* $n-1$ *indices* $2, 3, 4, \dots, n$ *we have*

$$1\mu\,.\,\mu\nu\,.\,\nu 1 = \mu 1\,.\,\nu\mu\,.\,1\nu$$

and

$$1\mu\,.\,\mu 1 = +.$$

(2). For shortness' sake, let us consider the equation of the 4th degree, viz.

$$\begin{vmatrix} a_1-x & a_2 & a_3 & a_4 \\ b_1 & b_2-x & b_3 & b_4 \\ c_1 & c_2 & c_3-x & c_4 \\ d_1 & d_2 & d_3 & d_4-x \end{vmatrix} = 0,$$

---

* Philos. Mag.

in which case we have to establish the reality of all the roots when it is given that

$$\begin{aligned} a_2b_3c_1 &= a_3b_1c_2, & a_2b_1 &= +, \\ a_2b_4d_1 &= a_4b_1d_2, & a_3c_1 &= +, \\ a_3c_4d_1 &= a_4c_1d_3, & a_4d_1 &= +, \end{aligned}$$

it being noted however that these three conditional equations imply another, viz.

$$b_3c_4d_2 = b_4c_2d_3,$$

and that, along with the three conditions as to sign, they also imply three others of the latter character, viz.

$$\begin{aligned} b_3c_2 &= +, \\ b_4d_2 &= +, \\ c_4d_3 &= +, \end{aligned}$$

so that, in fact, the determinant is given axisymmetric as to sign.

Denoting, then, the above equation of the 4^{th} degree by

$$f(x) = 0,$$

we have for the product of the two determinants $f(x)$ and $f(-x)$,

$$\begin{vmatrix} \dfrac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1} - x^2 & \dfrac{b_1, b_2, b_3, b_4}{a_1, b_1, c_1, d_1} & \dfrac{c_1, c_2, c_3, c_4}{a_1, b_1, c_1, d_1} & \dfrac{d_1, d_2, d_3, d_4}{a_1, b_1, c_1, d_1} \\ \dfrac{a_1, a_2, a_3, a_4}{a_2, b_2, c_2, d_2} & \dfrac{b_1, b_2, b_3, b_4}{a_2, b_2, c_2, d_2} - x^2 & \dfrac{c_1, c_2, c_3, c_4}{a_2, b_2, c_2, d_2} & \dfrac{d_1, d_2, d_3, d_4}{a_2, b_2, c_2, d_2} \\ \dfrac{a_1, a_2, a_3, a_4}{a_3, b_3, c_3, d_3} & \dfrac{b_1, b_2, b_3, b_4}{a_3, b_3, c_3, d_3} & \dfrac{c_1, c_2, c_3, c_4}{a_3, b_3, c_3, d_3} - x^2 & \dfrac{d_1, d_2, d_3, d_4}{a_3, b_3, c_3, d_3} \\ \dfrac{a_1, a_2, a_3, a_4}{a_4, b_4, c_4, d_4} & \dfrac{b_1, b_2, b_3, b_4}{a_4, b_4, c_4, d_4} & \dfrac{c_1, c_2, c_3, c_4}{a_4, b_4, c_4, d_4} & \dfrac{d_1, d_2, d_3, d_4}{a_4, b_4, c_4, d_4} - x^2 \end{vmatrix}$$

where $\dfrac{a, b, c, d}{m, n, p, q}$ is used for shortness' sake to denote the bipartite

$$(a, b, c, d \between m, n, p, q),$$

i. e. $am + bn + cp + dq$. Expanding this determinant according to descending powers of x^2, we obtain

$$\begin{aligned}
&x^8 \\
&-x^6\left\{\frac{a_1 a_2 a_3 a_4}{a_1 b_1 c_1 d_1}+\frac{b_1 b_2 b_3 b_4}{a_2 b_2 c_2 d_2}+\frac{c_1 c_2 c_3 c_4}{a_3 b_3 c_3 d_3}+\frac{d_1 d_2 d_3 d_4}{a_4 b_4 c_4 d_4}\right\} \\
&+x^4\left\{\begin{array}{l}
\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}\cdot\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{vmatrix}+\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}\cdot\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \\
+\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}\cdot\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}+\begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}\cdot\begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \\
+\begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}\cdot\begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}+\begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}\cdot\begin{vmatrix} a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}
\end{array}\right\} \\
&-x^2\left\{\begin{array}{l}
\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}\cdot\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}+\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}\cdot\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \\
\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}\cdot\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}+\begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}\cdot\begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}
\end{array}\right\} \\
&+\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}\cdot\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix};
\end{aligned}$$

and all that remains is to show that the coefficients of this expansion are all positive.

Now the coefficient of $-x^6$ is

$$\begin{aligned}
&a_1^2+a_2b_1+a_3c_1+a_4d_1 \\
+\,&b_1a_2+b_2^2+b_3c_2+b_4d_2 \\
+\,&c_1a_3+c_2b_3+c_3^2+c_4d_3 \\
+\,&d_1a_4+d_2b_4+d_3c_4+d_4^2,
\end{aligned}$$

and every individual term of it is seen to be positive because of the sign-conditions.

The coefficient of x^4 is the axisymmetric square array of 36 terms:

$$\begin{array}{llllll}
|a_1 b_2|^2 & +|a_1 b_3||a_1 c_2| & +|a_1 b_4||a_1 d_2| & +|a_2 b_3||b_1 c_2| & +|a_2 b_4||b_1 d_2| & +|a_3 b_4||c_1 d_2| \\
+|a_1 c_2||a_1 b_3| & +|a_1 c_3|^2 & +|a_1 c_4||a_1 d_3| & +|a_2 c_3||b_1 c_3| & +|a_2 c_4||b_1 d_3| & +|a_3 c_4||c_1 d_3| \\
+|a_1 d_2||a_1 b_4| & +|a_1 d_3||a_1 c_4| & +|a_1 d_4|^2 & +|a_2 d_3||b_1 c_4| & +|a_2 d_4||b_1 d_4| & +|a_3 d_4||c_1 d_4| \\
+|b_1 c_2||a_2 b_3| & +|b_1 c_3||a_2 c_3| & +|b_1 c_4||a_2 d_3| & +|b_2 c_3|^2 & +|b_2 c_4||b_2 d_3| & +|b_3 c_4||c_2 d_3| \\
+|b_1 d_2||a_2 b_4| & +|b_1 d_3||a_2 c_4| & +|b_1 d_4||a_2 d_4| & +|b_2 d_3||b_2 c_4| & +|b_2 d_4|^2 & +|b_3 d_4||c_2 d_4| \\
+|c_1 d_2||a_3 b_4| & +|c_1 d_3||a_3 c_4| & +|c_1 d_4||a_3 d_4| & +|c_2 d_3||b_3 c_4| & +|c_2 d_4|||\,b_3 d_4| & +|c_3 d_4|^2
\end{array}$$

where six of the terms are clearly positive and fifteen require examination. Taking the first of the latter, viz. $|a_1 b_3| . |a_1 c_2|$ and multiplying and dividing by b_3, we see that it

$$= |a_1 b_3| . (a_1 c_2 b_3 - a_2 c_1 b_3) \div b_3,$$

which, on account of the conditional equation $a_2 b_3 c_1 = a_3 b_1 c_2$,

$$\begin{aligned} &= |a_1 b_3| . (a_1 b_3 - a_3 b_1) c_2 \div b_3, \\ &= |a_1 b_3|^2 . \frac{c_2}{b_3}, \end{aligned}$$

and therefore is positive because b_3 and c_2 are given alike in sign. The same mode of proof suffices for all the others except the three in the second diagonal of the square, in which cases the introduced multiplier is the product of *two* elements. For example,

$$\begin{aligned} |a_3 b_4| . |c_1 d_2| &= |a_3 b_4| . (a_3 b_4 c_1 d_2 - a_3 b_4 c_2 d_1) \div a_3 b_4, \\ &= |a_3 b_4| . (a_3 b_4 c_1 d_2 - a_4 b_3 c_1 d_2) \div a_3 b_4, \\ &= |a_3 b_4|^2 . \frac{c_1 d_2}{a_3 b_4} \\ &= + \end{aligned}$$

because a_3 and c_1 are alike in sign, and also b_4 and d_2.

The coefficient of $-x^2$ is the axisymmetric square array of sixteen terms,

$$\begin{array}{llll} |a_1 b_2 c_3|^2 & + |a_1 b_2 c_4|\,|a_1 b_2 d_3| & + |a_1 b_3 c_4|\,|a_1 c_2 d_3| & + |a_2 b_3 c_4|\,|b_1 c_2 d_3| \\ + |a_1 b_2 d_3|\,|a_1 b_2 c_4| & + |a_1 b_2 d_4|^2 & + |a_1 b_3 d_4|\,|a_1 c_2 d_4| & + |a_2 b_3 d_4|\,|b_1 c_2 d_4| \\ + |a_1 c_2 d_3|\,|a_1 b_3 c_4| & + |a_1 c_2 d_4|\,|a_1 b_3 d_4| & + |a_1 c_3 d_4|^2 & + |a_2 c_3 d_4|\,|b_1 c_3 d_4| \\ + |b_1 c_2 d_3|\,|a_2 b_3 c_4| & + |b_1 c_2 d_4|\,|a_2 b_3 d_4| & + |b_1 c_3 d_4|\,|a_2 c_3 d_4| & + |b_2 c_3 d_4|^2 \end{array}$$

where four of the terms are clearly positive and six require examination. Taking the first of the latter, viz. $|a_1 b_2 c_4|\,|a_1 b_2 d_3|$ and multiplying and dividing by c_4, we obtain

$$|a_1 b_2 c_4| . \{ |a_1 b_2| d_3 c_4 - |a_1 d_2| b_3 c_4 + |b_1 d_2| a_3 c_4 \} \div c_4$$

which

$$\begin{aligned} &= |a_1 b_2 c_4| . \left\{ |a_1 b_2| d_3 c_4 - |a_1 b_4| . \frac{d_2}{b_4} b_3 c_4 + |a_2 b_4| \frac{d_1}{a_4} . a_3 c_4 \right\} \div c_4, \\ &= |a_1 b_2 c_4| . \{ |a_1 b_2| d_3 c_4 - |a_1 b_4| c_2 d_3 + |a_2 b_4| c_1 d_3 \} \div c_4, \\ &= |a_1 b_2 c_4| . |a_1 b_2 c_4| . \frac{d_3}{c_4} \\ &= + \end{aligned}$$

because c_4 and d_3 are alike in sign. For the five other terms, exactly the same mode of proof suffices.

Lastly, the coefficient of x^0 is manifestly $|a_1 b_2 c_3 d_4|^2$.

The theorem is thus established.

(3). There are several results obtained in the foregoing which are well worth fuller notice.

In the first place, it should be observed that not only is the coefficient of each power of x positive, but it is shown to be so by reason of the fact that every term of it is positive, exactly as in Sylvester's original case.

Secondly, it should be noted that in dealing with the coefficient of x^4 we proved that each minor of the 2^d order bears a simple ratio to its conjugate minor, viz.

$$\begin{array}{llll}
|a_1 b_3| : |a_1 c_2| & :: b_3 & : c_2 & , \\
|a_1 b_4| : |a\ d_2| & :: b_4 & : d_2 & , \\
|a_2 b_3| : |b_1 c_2| & :: a_3 & : c_1 & , \\
|a_2 b_4| : |b_1 d_2| & :: a_4 & : d_1 & , \\
|a_3 b_4| : |c_1 d_2| & :: a_3 b_4 & : c_1 d_2 & , \\
|a_1 c_4| : |a_1 d_3| & :: c_4 & : d_3 & , \\
|a_2 c_3| : |b_1 c_3| & :: a_2 & : b_1 & , \\
|a_2 c_4| : |b_1 d_3| & :: a_2 c_4 & : b_1 d_3 & , \\
|a_3 c_4| : |c_1 d_3| & :: a_4 & : d_1 & , \\
|a_2 d_3| : |b_1 c_4| & :: a_2 d_3 & : b_1 c_4 & , \\
|a_2 d_4| : |b_1 d_4| & :: a_2 & : b_1 & , \\
|a_3 d_4| : |c_1 d_4| & :: a_3 & : c_1 & , \\
|b_2 c_4| : |b_2 d_3| & :: c_4 & : d_3 & , \\
|b_3 c_4| : |c_2 d_3| & :: b_4 & : d_2 & , \\
|b_3 d_4| : |c_2 d_4| & :: b_3 & : c_2 & .
\end{array}$$

Thirdly, in dealing with the coefficient of x^2 it was shown that each minor of the 3^d order bears a similarly simple ratio to its conjugate minor, viz.

$$\begin{array}{l}
|a_1 b_2 c_4| : |a_1 b_2 d_3| :: c_4 : d_3 , \\
|a_1 b_3 c_4| : |a_1 c_2 d_3| :: b_4 : d_2 , \\
|a_2 b_3 c_4| : |b_1 c_2 d_3| :: a_4 : d_1 , \\
|a_1 b_3 d_4| : |a_1 c_2 d_4| :: b_3 : c_2 , \\
|a_2 b_3 d_4| : |b_1 c_2 d_4| :: a_3 : c_1 , \\
|a_2 c_3 d_4| : |b_1 c_3 d_4| :: a_2 : b_1 .
\end{array}$$

(4). A little examination suffices to make evident that only six ratios are involved in all these proportions, viz. the ratio of each non-diagonal element to its conjugate. All the proportions except three can thus be combined as follows:

$$\begin{aligned}
\frac{a_2}{b_1} &= \frac{|a_2\,c_3|}{|b_1\,c_3|} = \frac{|a_2\,d_4|}{|b_1\,d_4|} = \frac{|a_2\,c_3\,d_4|}{|b_1\,c_3\,d_4|}\,,\\
\frac{a_3}{c_1} &= \frac{|a_3\,b_2|}{|c_1\,b_2|} = \frac{|a_3\,d_4|}{|c_1\,d_4|} = \frac{|a_3\,b_2\,d_4|}{|c_1\,b_2\,d_4|}\,,\\
\frac{a_4}{d_1} &= \frac{|a_4\,b_2|}{|d_1\,b_2|} = \frac{|a_4\,c_3|}{|d_1\,c_3|} = \frac{|a_4\,b_2\,c_3|}{|d_1\,b_2\,c_3|}\,,\\
\frac{b_3}{c_2} &= \frac{|b_3\,a_1|}{|c_2\,a_1|} = \frac{|b_3\,d_4|}{|c_2\,d_4|} = \frac{|b_3\,a_1\,d_4|}{|c_2\,a_1\,d_4|}\,,\\
\frac{b_4}{d_2} &= \frac{|b_4\,a_1|}{|d_2\,a_1|} = \frac{|b_4\,c_3|}{|d_2\,c_3|} = \frac{|b_4\,a_1\,c_3|}{|d_2\,a_1\,c_3|}\,,\\
\frac{c_4}{d_3} &= \frac{|c_4\,a_1|}{|d_3\,a_1|} = \frac{|c_4\,b_2|}{|d_3\,b_2|} = \frac{|c_4\,a_1\,b_2|}{|d_3\,a_1\,b_2|}\,.
\end{aligned}$$

It therefore appears that we can formulate the following general theorem: *If the determinant of the* 4*th order* $|a_1\,b_2\,c_3\,d_4|$ *fulfil the specified conditions, the ratio of any non-diagonal element* pq *to its conjugate* qp

$$= \frac{|pq\,.\,rr|}{|qp\,.\,rr|} = \frac{|pq\,.\,ss|}{|qp\,.\,ss|} = \frac{|pq\,.\,rr\,.\,ss|}{|qp\,.\,rr\,.\,ss|}\,.$$

In the case of the three excepted proportions the ratios which occur are compounded of these six, viz.

$$\begin{aligned}
\frac{a_2\,c_4}{b_1\,d_3} &= \frac{a_4\,c_2}{b_3\,d_1} = \frac{|a_2\,c_4|}{|b_1 d_3|}\,,\\
\frac{a_3\,b_4}{c_1\,d_2} &= \frac{a_4\,b_3}{c_2\,d_1} = \frac{|a_3\,b_4|}{|c_1\,d_2|}\,,\\
\frac{a_2\,d_3}{b_1\,c_4} &= \frac{a_3\,d_2}{b_4\,c_1} = \frac{|a_2 d_3|}{|b_1\,c_4|}\,.
\end{aligned}$$

(5). Another important property of the determinant $|a_1\,b_2\,c_3\,d_4|$ is that *every term is equal to its conjugate.*

To establish this we have to prove seven identities, for of the total twenty-four terms ten are known* to be self-conjugate. The first four

---

* Proc. Roy. Soc. Edinburgh, XVII, pp. 7–13.

$$a_2 b_3 c_1 d_4 = a_3 b_1 c_2 d_4,$$
$$a_2 b_4 c_3 d_1 = a_4 b_1 c_3 d_2,$$
$$a_3 b_2 c_4 d_1 = a_4 b_2 c_1 d_3,$$
$$a_1 b_3 c_4 d_2 = a_1 b_4 c_2 d_3,$$

are readily got from the conditional equations by multiplying by d_4, c_3, b_2, a_1 respectively. Each of the remaining three is got from a *pair* of the conditional equations by multiplication and division, thus—

from the 1st and 4th equations

$$a_2 b_3 c_1 . b_4 c_2 d_3 = a_3 b_1 c_2 . b_3 c_4 d_2,$$

and ∴ $$a_2 b_4 c_1 d_3 = a_3 b_1 c_4 d_2;$$

from the 2d and 4th equations

$$a_2 b_4 d_1 . b_3 c_4 d_2 = a_4 b_1 d_2 . b_4 c_2 d_3$$

and ∴ $$a_2 b_3 c_4 d_1 = a_4 b_1 c_2 d_3;$$

and from the 3d and 4th equations

$$a_3 c_4 d_1 . b_4 c_2 d_3 = a_4 c_1 d_3 . b_3 c_4 d_2$$

and ∴ $$a_3 b_4 c_2 d_1 = a_4 b_3 c_1 d_2.$$

(6). Finally, it should be noted that the conditions here given for the reality of the roots of $f(x) = 0$ are exactly the conditions for the determinant $f(x)$ being transformable by multiplication of rows and columns into an axisymmetric determinant in which x is involved in the same way as before the transformation.